Certain associated graded rings of 3-dimensional regular local rings are regular (Newton polyhedrons and Singularities)

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Certain associated graded rings of 3-dimensional regular local rings are regular

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This note is a preliminary version.

Introduction. The study of various blowing-ups is very important in the theory of singularities. In many case some blowing-up appears as the blowing-down of divisors of algebraic variety , and is understood naturally as a filtered blowing-up. From this point of view, one of most interesting results in this field is M. Kawakita's classification of a special divisorial contraction of dimension three [2]. In [2], Kawakita proved that every divisorial contraction to a smooth 3-dimensional point is a weighted blowing-up induced by certain weighting on a regular system of parameters of 3-dimensional regular local ring. It is natural to study his theorem from the theory of filtered blowing-ups, and this is my motivation for this talk.

In this paper, I will discuss the filtered blowing-up of singularities, and, by using special equi-singular deformation induced from a filtration on local ring, I show the following simple assertion,

Theorem. 1 *Let $A\cong \mathbf{C}\{x_1,x_2,x_3\}$ and $F=\{F^k\}_{k\geq 0}$ be a filtration on A such that $gr_FA=\oplus_{k\geq 0}F^k/F^{k+1}$ is an integral domain with isolated singularity. Then gr_FA is regular, i.e., $gr_FA\cong \mathbf{C}[y_1,y_2,y_3]$.*

In this paper, a filtration F on the local ring (A,m) is; $F=\{F^k\}$; a decreasing sequence of ideals $F^k\subset A$ such that $F^0(A)=A, m\supset F^1, F^k=A(k\leq -1), F^kF^l\subset F^{k+l}(\forall k,l)$ and $\mathcal{R}=\oplus_{k\geq 0}F^kT^k\subset A[T]$ is a finitely generated A-algebra, where T is an indeterminate. There is an integer N such that the relation $F^{kN}=F^N\cdots F^N$ for all $k\geq 0$, and we assume that F^N is m-primary. We denote $G=gr_F(A)$ and remark that $G=\mathcal{R}'/T^{-1}\mathcal{R}'$, where $\mathcal{R}'=\oplus_{k\in Z}F^kT^k$ is the extended Rees algebra.

Theorem 1 is shown as a special case of the following more general results.

Theorem 2. *Let (V,p) be a normal d-dimensional isolated terminal singularity of index r (resp. canonical, resp. log terminal, resp. log canonical), and $F=\{F^k\}$ be a filtration on $A=O_{V,p}$ such that $G=gr_FA$ is an integral domain with isolated singularity. Then*

(1) G is normal and terminal singularity of index r (resp. canonical, resp. log terminal, resp. log canonical).

(2) There is a filtration $F_B=\{F_B^k\}$ on the canonical cover (the index one cover) $B=\oplus_{m=0}^{k-1}\omega_A^{[m]}$ such that $G_B=gr_{F_B}B\cong$ the canonical cover of G and there exists an integer $M\geq 1$ such that the relations $F_B^{kM}\cap A=F^k\subset A$ for $k\geq 0$ and $(gr_{F_B\cap A}(A))^{(M)}=gr_F(A)$ hold.

(3) If $d=3$ and (V,p) is terminal, then the relation $e(m_B,B)=e((G_B)_+,G_B)(=1,2)$ holds.

We have a corollary as follows:

Corollary 3. *(V,p): 3-dimensional cyclic terminal and F: as above, then $gr_F(A)^\wedge \cong A^\wedge$.*

As the case of index one, we obtain Theorem 1 from Corollary 3. Here recall that every isolated quotient singularity of dimension not less than three is rigid.

In general, if we consider a filtration induced from a divisorial contraction, the associated graded ring is not necessary an integral domain with isolated singularity ([1,3]).

§1. Sketch of proof of Theorem 2.

We assume that there is no $N \geq 2$ such that $G^{(N)} = G$, where $G^{(N)}$ is defined by $G^{(N)} = \oplus_{k \geq 0} G_{kN} \subset G$.

Step 1. Let $\psi : X = \mathrm{Proj}(\mathcal{R}) \to V = \mathrm{Spec}A$ be the filtered blowing-up by F with $E = \mathrm{Proj}(G)$. We obtain the relation $F^k = \phi_*(O_X(-kE))$ for $k \in \mathbf{Z}$. (cf [6, §2]).

Proof. Since G is an integral domain and $V = \mathrm{Spec}A$ is normal, we can easily see that $\mathcal{R}' = \oplus_{k \in Z} F^k T^k \subset A[T, T^{-1}]$ is a normal domain.

This claim is shown as follows: We have $G = \mathcal{R}'/u\mathcal{R}'$, where $u = T^{-1} \ni \mathcal{R}'_{-1}$. If $P \in V(u) \subset \mathrm{Spec}(\mathcal{R}')$, then $G_P \cong \mathcal{R}'_P/u\mathcal{R}'_P$ satisfies the conditions R_0 and S_1, hence $\mathcal{R}'_P$ is normal. Further, if $P \notin V(u)$, then we obtain the relations $\mathcal{R}'_P = (\mathcal{R}'_T)_P = A[T, T^{-1}]_P$ which is normal.

By the assumption that $\mathcal{R}$ is a finitely generated A-algebra, there is a positive integer $N > 0$ such that $F^{kN} = F^N \cdots F^N$, for $k \geq 0$, i.e., $\mathcal{R}^{(N)} = A[F^N T^N]$. Here ψ is the blowing-up with center F^N and $F^{kN} = \psi_*(F^{kN} O_X) = \psi_*(O_X(kN))$. Since $Q(G)$ has a homogeneous element of degree 1, we have $O_X(k) = (O_X(1)^{\otimes k})^{**}$ for $\forall k \in \mathbf{Z}$. We have $O_X(1) = O_X(-E)$, hence $O_X(N) = O_X(-NE)$. Since G is an integral domain, $\{F^k\}$ defines a valuation V on $Q(A)$ such that $F^k = \{x \in Q(A) \mid V(x) \geq k\}$. Further $\{F^{kN}\}$ defines the valuation V' on $Q(A)$ as $F^{kN} = \{x \in Q(A) \mid V_E(x) \geq kN\}$ where $V_E(x) = ord_E(x)$ on X. Therefore $F^k = \{x \in Q(A) \mid V_E(x) \geq k\}$ for $\forall k \in \mathbf{Z}$.

Step 2. X has only cyclic quotient singularities, in particular X has only log terminal singularities.

Proof. (cf §5 [6]). For $P \in E = \mathrm{Proj}(G) \subset X = \mathrm{Proj}(\mathcal{R})$, there exists $f \in F^d - F^{d+1}$, with $P \in V_+(f^*)$, where $f^* = fT^d \in \mathcal{R}_d$. Here we denotes $\bar{f}T^d \in G_d$. Now $\mathcal{R}_{f^*} = \oplus_{k \in Z} (\mathcal{R}_{f^*})_k$ is a regular ring. This is shown as follows: We see that $(\mathcal{R}_{f^*})_{(T^{-1})^{-1}} = A_f[T, T^{-1}]$ is regular and that $\mathcal{R}_{f^*}/T^{-1}\mathcal{R}_{f^*} \doteq \mathcal{R}'_{f^*}/T^{-1}\mathcal{R}'_{f^*} = G_{\bar{f}}$ is regular. Hence so is $\mathcal{R}_{f^*}$.

Now let $B = (\mathcal{R}_{f_*})_P = \oplus_{k \in Z}((\mathcal{R}_{f_*})_P)_k$ and $t \in B$ be a homogeneous unit of the minimal degree $N(P)$. Let $C = B/t - 1$. Then, by [6,§5], C is a regular local ring. Here $((\mathcal{R}_{f_*})_P)_0$ is a finite direct summand of C.

Step 3. (The log canonical condition of A implies that) G is normal.

Proof. Let $\omega_0 \in \omega_A^{[r]}$ be a generator at p as $\omega_A^{[r]} = A \cdot \omega_0$. We define the integer

a' by the relation $div_X(\omega_0) = -(r+a')E$ on X. That is $\omega_X^{[r]} \cong O_X(-(r+a')E)$ or $K_X = \psi^*(K_V) - (1+\frac{a'}{r})E$. Since A is log canonical, we have $a' \leq 0$. We will show the following.

Claim. $R^1\psi_*(O_X(-mE)) = 0$ for $m \geq 1$, $(m \in \mathbf{Z})$.

Proof of the claim. We have the relation

$$\begin{aligned} O_X(-mE) &\cong \omega_X((r-1)K_X + (r+a')E - mE)) \\ &\cong \omega_X((r-1)(K_X+E) - (m-1-a')E). \end{aligned}$$

Further $(r-1)(K_X+E)-(m-1-a')E$ is relatively numerically equivalent to $-\frac{r-1}{r}a'E-(m-1-a')E = -(m-1-\frac{a'}{r})E$ with respect to ψ. Since $-E$ is relatively ψ-ample, $(r-1)(K_X+E)-(m-1-a')E$ is ψ-nef. Hence by the vanishing theorem of Grauert-Riemenschneider,Kawamata-Viehweg, we obtain the claim.

Here we have the exact sequence

$$0 \to O_X(-(k+1)E) \to O_X(-kE) \to O_E(k) \to 0$$

for $k \in \mathbf{Z}$. By the claim, we obtain the following exact sequence

$$\begin{aligned} 0 &\to F^{k+1} = \psi_*(O_X(-(k+1)E)) \to F^k = \psi_*(O_X(-kE)) \to H^0(O_E(k)) \\ &\to R^1\psi_*(O_X(-(k+1)E)) = 0 \end{aligned}$$

for $k \geq 0$.

We have

$$0 \to H^0_{G_+}(G) \to G \to \oplus_{k\in\mathbf{Z}} H^0(O_E(k)) \to H^1_{G_+}(G) \to 0.$$

Since G is an integral domain, $H^0_{G_+}(G) = 0$. Further $\oplus_{k\in\mathbf{Z}}H^0(O_E(k)) = \Gamma_*(G)$ is normal. This is shown as follows: Let $\bar{G}$ be the normalization of G in $Q(G)$. Since G has only isolated singularity , $\bar{G}/G$ has finite length. Hence on $E = \mathrm{Proj}(G)$, we have the relation $\tilde{\bar{G}}(k) = \tilde{G}(k)$. By Demazure, with $T \in Q(\bar{G})_1$, there exists $D \in Div(E)\otimes\mathbf{Q}$ as follows; $\tilde{\bar{G}}(k) = O_E(kD)T^k$, for $k \in \mathbf{Z}$. Hence $\Gamma_*(G) = R(E,D) = \bar{G}$.

Therefore, we obtain the relation $H^1_{G_+}(G)_k = 0$ for $k \leq -1$. And the relation $H^1_{G_+}(G) = 0$ follows.

Step 4. We will discuss the log terminal property of $G = R(E,D)$ under the assumption that A is log terminal of index r.

We have the following.

Lemma [8]. *Let us assume the conditions that G is an integral domain where Spec(G)$-V(G_+)$ is normal Gorenstein and that Spec(A) $- V(m)$ is Gorenstein. Then the following relations hold.*

$$\frac{\omega_X^{[m]}(mE-\alpha E)}{\omega_X^{[m]}(mE-(\alpha+1)E)} \cong \omega_E^{[m]}(mD'+\alpha D) \;\; for\; m,\alpha \in \mathbf{Z}.$$

Here $O_E(k) = O_E(kD)T^k$ *as before, with* $D = \sum_{V\in Irr^1(X)} \frac{p_V}{q_V}V$ *with* $(p_V, q_V) = 1$,

$q_V \geq 1$ *and* $D' = \sum_{V \in Irr^1(X)} \frac{q_V - 1}{q_V} V$.

By the relation

$$\omega_X^{[r]}(rE - \alpha E) \cong O_X(-(a' + \alpha)E), \quad \text{for all } \alpha \in \mathbf{Z}$$

we obtain

$$\omega_E^{[r]}(rD' + \alpha D) \cong O_E((\alpha + a')D), \quad \text{for all } \alpha \in \mathbf{Z}$$

by Lemma. Hence $K_R^{[r]} = R(a')$ follows.

Here $\mathrm{Spec}(R) - V(R_+) = \mathrm{Spec}(G) - V(G_+)$ is regular, $G = R(E, D)$ is log terminal (resp. log canonical) if and only if $a' < 0$ (resp. $a' \leq 0$) by Theorem (2.5) and Theorem (2.8) of [7].

We will discuss the index of R. By Lemma, we have the following exact sequence.

$$0 \to \frac{T^m \omega_{\mathcal{R}'}^{[m]}}{T^{m-1}\omega_{\mathcal{R}'}^{[m]}} \to K_{R(E,D)}^{[m]} \to$$
$$\oplus_{k \in \mathbf{Z}} \operatorname{Ker}\left\{H^1(\omega_X^{[m]}(mE - (k+1)E)) \to H^1(\omega_X^{[m]}(mE - kE))\right\} \to 0 \text{ for } m \in \mathbf{Z}.$$

If there exist $r' > 1$ where the relation $K_{R(E,D)}^{[r']} = R(a")$ is satisfied for some integer $a" \in \mathbf{Z}$, we have the relation $\frac{a"}{r'} = \frac{a'}{r}$. We obtain $a" < 0$.

Here $(K_R^{[r']})_k = R_{k+a"}$, hence $(K_R^{[r']})_k = 0$ if $k \leq -1$.

For $k \geq 0$, we set $m = r' \geq 1$ and obtain the relations

$$\omega_X^{[m]}(mE - (k+1)E) = \omega_X((m-1)(K_X + E) - kE),$$

and

$$(m-1)(K_X + E) - kE \equiv -\left(-\frac{m-1}{r}a' + k\right)E.$$

This is ψ-nef, hence the following vanishing hold

$$H^1(\omega_X^{[m]}(mE - (k+1)E)) = 0 \text{ for } k \geq 0.$$

Hence $\frac{T^m \omega_{\mathcal{R}'}^{[m]}}{T^{m-1}\omega_{\mathcal{R}'}^{[m]}} \cong K_{R(E,D)}^{[m]}$ with $m = r'$. Hence $T^m \omega_{\mathcal{R}'}^{[m]}$ is locally principal along $V(T^{-1}) = \mathrm{Spec}(R(E, D)) \subset \mathrm{Spec}(\mathcal{R}')$. For $c \neq 0 \in \mathrm{Spec}\mathbf{C}[T^{-1}]$, it follows that $\omega_{\mathcal{R}'}^{[m]}/(T^{-1} - c)\omega_{\mathcal{R}'}^{[m]} = \cup_{k \in \mathbf{Z}} \psi_*(\omega_X^{[m]}(-kE)) = \omega_A^{[m]}$ is a principal $\mathcal{R}'/(T^{-1} - c)\mathcal{R}' = A$-module for same c.

Step 5. We will show : *The condition that A is a canonical (resp. terminal) singularity implies that G is also a canonical (resp. terminal) singularity.*

Proof. Let $\omega : \mathcal{V} = \mathrm{Spec}(\mathcal{R}') \to \mathrm{Spec}\mathbf{C}[T^{-1}] \cong \mathbf{C}$ with $\mathcal{V}_0 = \mathrm{Spec}G$, and $\mathcal{V}_c \cong V$ for $c \neq 0$. Let us introduce the filtration of ideals $\{F^l(\mathcal{R}')\}$ on $\mathcal{R}'$ by the following way: $F^l(\mathcal{R}') = \mathcal{R}' \mid_l \cdot \mathcal{R}' \subset \mathcal{R}'$, where $\mathcal{R}' = \oplus_{k \geq l} F^l T^l \subset \mathcal{R}'$ for $l \in \mathbf{Z}$. As is shown in [6]§5, we obtain the following diagram after the blowing-up of $\mathcal{V} = \mathrm{Spec}(\mathcal{R}')$ by this filtration.

$$\begin{array}{ccc} Y" = \mathrm{Proj}(\mathcal{R}_{\mathcal{F}}(\mathcal{R}')) & \xrightarrow{\xi} & \mathrm{Spec}(\mathcal{R}') = \mathcal{V} \\ \omega" \searrow & & \swarrow \omega \\ & \mathrm{Spec}\mathbf{C}[T^{-1}] & \end{array},$$

where $\omega"$ gives the filtered blowing-up for each fiber as follows: $\omega"_0 : Y"_0 \to \mathrm{Spec}(\mathcal{V}_0)$ is nothing but the graded blowing-up of Spec(G) and $\omega_c" : Y"_c \to \mathrm{Spec}(\mathcal{V}_c)$ is nothing but the blowing-up of Spec(A) by F for $c \neq 0 \in \mathbf{C}$. By J. Wahl [9], $\omega"$ is a locally trivial family under the assumption that Spec$(G) - V(G_+)$ is regular. Here $\mathcal{V}$ is an r-Gorenstein $d+1$-dimensional scheme and we have the following relation

$$K_{\mathcal{R}'}^{[r]} \cong \mathcal{R}'(a'+r).$$

There is a meromorphic r-ple $d+1$-form $\tilde{\Omega}_0$ of $\mathcal{R}'$ such that $\mathcal{R}' \to K_{\mathcal{R}'}^{[r]}; 1 \to \tilde{\Omega}_0$ gives an isomorphism. This induces the isomorphism

$$\omega_{Y"}^{[r]} = O_{Y"}(r+a')\xi^*(\tilde{\Omega}_0),$$

that is, we have the relation $div_{Y"}\tilde{\Omega}_0 = -(r+a')\mathbf{E}$, where the relation $\mathrm{Proj}gr_{\mathcal{F}}(\mathcal{R}') = \mathbf{E} \cong E \times \mathbf{C}$. Here $E = \mathrm{Proj}(G)$. Since $a' \leq -r$, $\xi^*(\tilde{\Omega}_0)$ is holomorphic on $Y"$. Hence $\mathrm{Res}_{(Y")_c}(\tilde{\Omega}_0)$ is a holomorphic r-ple d-form on $(Y")_c$ which does not vanishes on $(Y")_c - E$. Here $(Y")_c = X = \mathrm{Proj}(\mathcal{R})$ for $c \neq 0$, and $(Y")_c = \mathrm{Proj}(G^\natural) = C(E, D)$ for the case $c = 0$. Here $\mathrm{Res}_{(Z')_c}(\tilde{\Omega}_0)$ gives a generator of $\omega_{(Z')_c}^{[r]}$ for $c \in \mathbf{C} = \mathrm{Spec}(\mathbf{C}[T^{-1}])$.

We state the following claim.

Claim. There is a resolution of singularities $\beta : \tilde{Y"} \to Y"$ such that the natural induced map $\tilde{\omega"} : \tilde{Y"} \to \mathbf{C}$ is locally trivial along the fiber over $\{0\} = V(T^{-1})$:

$$\begin{array}{ccc} \tilde{Y"} & \xrightarrow{\beta} & Y" = \mathrm{Proj}(\mathcal{R}_{\mathcal{F}}(\mathcal{R}')) \\ \tilde{\omega"} \searrow & & \swarrow \omega" \\ & \mathrm{Spec}\mathbf{C}[T^{-1}] & \end{array}.$$

Let $\mathbf{F} \subset \tilde{Y"} \to C$ be the horizontal divisor of $\tilde{Y"}$ which is exceptional for $\beta : \tilde{Y"} \to Y"$. For $c \neq 0$, we have the relation:

$$\mathrm{Res}\,|_{\tilde{Y"}_c} \left(\beta^*(\tilde{\Omega}_0)\right) = \beta^* \left(\mathrm{Res}\,|_{Y"_c}\, \tilde{\Omega}_0\right).$$

Since (A, m) has only canonical singularities, this is holomorphic. Hence $\tilde{\Omega}_0$ is holomorphic on $\tilde{Y"}$. Therefore $\mathrm{Res}\,|_{\tilde{Y"}_0} \left(\beta^*(\tilde{\Omega}_0\right)$ is holomorphic.

Q.E.D. for the claim.

Step 6. Here we will introduce a filtration F_B on the local ring $B = \oplus_{k=0}^{r-1}\omega_A^{[k]}$ which has the desired properties as is claimed in Theorem 2.

By a tentative way, we set $F_B^k(\omega_A^{[m]}) \subset \omega_A^{[m]}$ as follows:

$$F_B^k(\omega_A^{[m]}) = \sum_{ma'+rh \geq k \cdot gcd(a',r)} \psi_* \left(\omega_X^{[m]}(mE - kE))\right) \subset \omega_A^{[m]},$$

and

$$F_B^k(B) = \oplus_{m=0}^{r-1} F_B^k(\omega_A^{[m]})U^m \subset B = \oplus_{m=0}^{r-1}\omega_A^{[m]}U^m.$$

The main point which we have to check here is the assertion that the associated graded ring of $gr_{F_B}B$ is nothing but the graded canonical cover $G = R(E, D)$. We can show this asserition by the following formula about graded cyclic covers which we will recall in the below.

Now, K_G is a $\mathbf{Q}$−Cartier divisor of index r and there exists $\varphi \in k(X)$ such that $rK_E - a'D = div_X(\varphi)$.

Corollary (1.7.1) of [7]. *Let $S = S(R, K_R, \varphi T^{a'})$ be the normal graded cyclic r-cover of $R = R(X, D)$ as described in [7]. Then the Pinkham-Demazure construction S with respect to $\tilde{T} = T^{\beta}u^{\alpha}$ with $\alpha a' + \beta r = s(= (r, a'))$ is given by $S = R(F, \tilde{D})$ as follows:*
(1) F is the cyclic cover of E given by

$$\rho : F = Spec_E(\bigoplus_{l=0}^{s-1} O_E(l\left(\frac{r}{s}(K_X + D') - \frac{a'}{s}D\right))) \to E.$$

(2) $\tilde{D} = \rho^*\{\alpha(K_X + D') + \beta D\}$.
(3) We obtain the relation $K_S = S(\frac{a'}{s})$.

By using Lemma B and the above theorem we can check the assertion. The details are left to the readers.

Further we obtain the following relations;

$$F_B^k \cap A = F_B^k(\omega_Z^{[0]}) = \sum_{h \geq k\frac{gcd(a',r)}{r}} \psi_*(O_X(-hE)) = F^{[k\frac{gcd(a',r)}{r}]}.$$

Step 7. Now we assume that $d = \dim A = 3$ and that (V, p) is a terminal singularity of index r. Then so is $gr_F(A) = R(E, D)$. Since $gr_{F_B}(B)$ is the graded canonical cover of $gr_F(A)$, $gr_{F_B}(B)$ is a terminal 3-dimensional singularity of index one, hence is regular or conpund Du Val singularity. In particular, $gr_{F_B}(B)$ is a hypersurface isolated singularity by M. Reid [4].

We have the following results on multiplicities of filtered rings;

Lemma [5]. Let $P(G_B, \lambda) = \sum_{k \geq 0} l((G_B)_k)\lambda^k \in \mathbf{Z}[[\lambda]]$ and $x_1, \ldots, x_x \in (G_B)_+$ be a homogeneous minimal generator with $\deg x_1 \leq \deg x_2 \leq \ldots \leq \deg x_s$. Then we have the followings.

(1) $\deg x_1 \cdot \deg x_2 \cdots \deg x_d \lim_{\lambda \to 1}(1 - \lambda)^d P(G_B, \lambda) \leq e(m_B, B) \leq e((G_B)_+, G_B)$. Hence, if $e((G_B)_+, G_B)$ equals the round up of the rational number $\deg x_1 \cdot \deg x_2 \cdots \deg x_d \lim_{\lambda \to 1}(1 - \lambda)^d P(G_B, \lambda)$, then we have the equality $e(m_B, B) = e((G_B)_+, G_B)$.

(2) If G_B is a hypersurface isolated singularity which is defined by a quasi-homogeneous polynomial of type $(\deg x_1, \ldots, \deg x_{d+1}; h)$, then $\deg x_1 \cdot \deg x_2 \cdots \deg x_d \lim_{\lambda \to 1}(1 - \lambda)^d P(G_B, \lambda) = \dfrac{h}{\deg x_{d+1}}$ and $e((G_B)_+, G_B)$ equals to the round up of the rational number $\dfrac{h}{\deg x_{d+1}}$.

Hence we obtain the relation $e(m_B, B) = e((G_B)_+, G_B)(= 1, \text{or} 2)$.

This completes the proof of Theorem 2.

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